6. Representable Matroids

6.4. Constructing representations for matroids

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Based on *Matroid Theory* by James Oxley

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Overview

Strategy of this section.

- Start from a standard form $[I_r \mid D]$.
- Keep only the zero/nonzero pattern, obtaining $D^{\#}$.
- Use determinants to detect which exchanged sets are bases.
- Use row/column scaling and the graph $G(D^{\#})$ to determine which entries of D can be chosen freely.
- Apply the method to F_7, F_7^-, and the family $P_8, P_8^-, P_8^=$.

Message. This section is not just about deciding representability; it is about *building* representations explicitly.

B-Fundamental-Circuit Incidence Matrix

Standard representation. Let $[I_r \mid D]$ be a standard representation of a rank-r matroid M, with

$$B = \{e_1, \ldots, e_r\}.$$

Fundamental circuits. For each $k \in \{r+1, \ldots, n\}$,

$$C(e_k, B) = \{e_k\} \cup \{e_i \in B : D_{ik} \neq 0\}.$$

Let $D^{\#}$ be obtained from D by replacing each nonzero entry by 1. Then each column of $D^{\#}$ is the incidence vector of $C(e_k, B) - e_k$.

B-Fundamental-Circuit Incidence Matrix

Hence, $D^{\#}$ records the fundamental circuits relative to B. The matrix $D^{\#}$ is called the B-**fundamental-circuit incidence matrix** of M, and $[I_r \mid D^{\#}]$ is called a **partial representation** of M.

Why this matters. When constructing a representation, we first know *where* the nonzero entries must be; the remaining problem is to determine *which field values* those nonzero entries should have.

B-Fundamental-Circuit Incidence Matrix

Proposition 6.4.1

Let $[I_r \mid D_1]$ and $[I_r \mid D_2]$ be matrices over the fields K_1 and K_2 with the columns of each labelled, in order, by $e_1, e_2, \ldots, e_n$. If the identity map on $\{e_1, e_2, \ldots, e_n\}$ is an isomorphism from $M[I_r \mid D_1]$ to $M[I_r \mid D_2]$, then

$$D_1^{\#} = D_2^{\#}.$$

Interpretation. Although the actual nonzero entries of D may depend on the field, the zero/nonzero pattern of D is determined by the fundamental circuits relative to the basis B.

Consequence for binary matroids. For binary matroids, there is no freedom in the nonzero pattern over $\mathrm{GF}(2)$; this is the first hint of uniqueness of binary representations.

B-Fundamental-Circuit Incidence Matrix

Proposition 6.4.2

Let M be a matroid on a set E and B be a basis of M. If X is the B-fundamental-circuit incidence matrix of M, then X^T is the $(E - B)$-fundamental-circuit incidence matrix of M^*.

Idea of the proof. Passing to the dual matroid exchanges the roles of B and $E - B$. Accordingly, the fundamental-circuit incidence matrix for M^* is obtained simply by transposing the matrix for M.

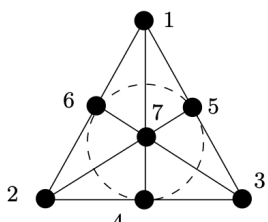
The Fano and Non-Fano Matroids

Why these examples? The matroids F_7 and F_7^- are the first major examples in this section where the same partial representation leads to different matroids over different fields.

The Fano and Non-Fano Matroids

Fano matroid F_7. The Fano matroid is the rank-3 matroid represented by the seven points of the projective plane over $\mathrm{GF}(2)$.

Non-Fano matroid F_7^-. The non-Fano matroid is obtained by changing one line of F_7: in F_7, $\{4,5,6\}$ is a circuit, while in F_7^- it is independent.



(a)

The Fano and Non-Fano Matroids

Common fundamental-circuit pattern. For both matroids, if

$$B = \{1, 2, 3\},$$

then the B-fundamental-circuit incidence matrix is

$$X = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}.$$

Key point. The two matroids have the same partial representation $[I_3 \mid X]$. So the difference between F_7 and F_7^- is not in the zero/nonzero pattern, but in how that matrix behaves over fields of different characteristic.

The Fano and Non-Fano Matroids

Lemma 6.4.4

Let $A = [I_3|X]$ where X is the matrix in Figure 6.12(b). View A over a field F. If the characteristic of F is two, then

$$M[A] = F_7,$$

while if the characteristic of F is not two, then

$$M[A] = F_7^-.$$

Interpretation. The same matrix $[I_3|X]$ represents different matroids depending on the characteristic of the field.

If $\mathrm{char}(F) = 2$, then $\{4, 5, 6\}$ is dependent, giving F_7. If $\mathrm{char}(F) \neq 2$, then $\{4, 5, 6\}$ is independent, giving F_7^-.

Determinant Agreement of Submatrices

Proposition 6.4.5

Let $[I_r|D_1]$ and $[I_r|D_2]$ be $r \times n$ matrices over the fields K_1 and K_2 with the columns of each being labelled, in order, by $e_1, e_2, \ldots, e_n$. Then $M[I_r|D_1]$ and $M[I_r|D_2]$ are isomorphic under the identity map on $\{e_1, e_2, \ldots, e_n\}$ if and only if, whenever D_1' and D_2' are corresponding square submatrices of D_1 and D_2, respectively, $\det D_1' = 0$ exactly when $\det D_2' = 0$.

Why this proposition? Proposition 6.4.1 only records which entries of D are zero or nonzero. Equivalently, it only uses the vanishing pattern of 1×1 minors. Proposition 6.4.5 extends this idea to all square submatrices.

Determinant Agreement of Submatrices

Interpretation. Two standard representations define the same matroid exactly when corresponding square submatrices are singular in the same places.
Thus, the matroid depends not on the actual determinant values, but only on which determinants vanish.

Idea of the proof. For a square submatrix D', the condition $\det D' \neq 0$ means that the corresponding set

$$(B - \{\text{chosen rows}\}) \cup \{\text{chosen columns}\}$$

is a basis. Therefore $\det D'_1 \neq 0$ if and only if the corresponding exchanged set is a basis of $M[I_r|D_1]$. Since the identity map is an isomorphism, the same set is a basis of $M[I_r|D_2]$, which implies $\det D'_2 \neq 0$.

From Partial Representations to Actual Representations

So far.

- $D^{\#}$ tells us the zero/nonzero pattern.
- Determinants tell us which exchanged sets are bases.

Next question. Suppose the partial representation $[I_r \mid D^{\#}]$ is already known. How much freedom remains in choosing the actual nonzero entries of D?

Answer. Row/column scaling and the bipartite graph $G(D^{\#})$ measure exactly that freedom.

From Representations to Graphs

Goal. Given the partial representation

$$[I_r \mid D^{\#}],$$

we want to determine the actual nonzero values in D.

Scaling idea. By row and column scaling, many entries of D can be normalized to 1.

Associated graph. From $D^{\#}$, define a bipartite graph $G(D^{\#})$:

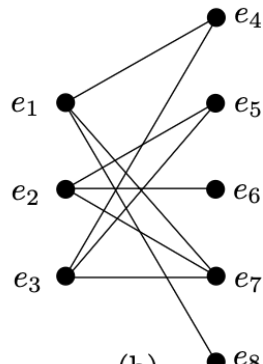
- Row vertices: $e_1, \ldots, e_r$
- Column vertices: $e_{r+1}, \ldots, e_n$
- Edge $e_i e_j$ exists iff the (i, j)-entry of $D^{\#}$ is 1

Thus, nonzero entries of D correspond to edges of $G(D^{\#})$.

From Representations to Graphs

$$
\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array}
\begin{array}{c}
\begin{array}{ccccc} e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\
\left[\begin{array}{ccccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right]
\end{array}
$$

(a)



(b)

From Representations to Graphs

Theorem 6.4.7

For a field F, let the $r \times n$ matrix $[I_r|D_1]$ be an F-representation of a matroid M. Let $\{b_1, b_2, \ldots, b_k\}$ be a basis of the cycle matroid of $G(D_1^{\#})$. Then

$$k = n - \omega(G(D_1^{\#})).$$

Moreover, if $(\theta_1, \theta_2, \ldots, \theta_k)$ is an ordered k-tuple of nonzero elements of F, then M has a unique F-representation $[I_r|D_2]$ projectively equivalent to $[I_r|D_1]$ such that the entries corresponding to b_i are θ_i.

From Representations to Graphs

Interpretation. A basis of the cycle matroid of $G(D^{\#})$ specifies the entries of D that may be chosen independently by scaling. Within a fixed projective-equivalence class, fixing those entries uniquely determines the remaining entries.

So Theorem 6.4.7 turns the construction problem into a graph-theoretic one: the entries normalized by scaling correspond to a spanning forest of $G(D^{\#})$.

From Representations to Graphs

Lemma 6.4.9

If $M \in \{F_7, F_7^-\}$ and M is F-representable, then

$$M = M[I_3|X]$$

where X is the matrix in Figure 6.12(b).

Interpretation. To determine representability of F_7 or F_7^-, it is enough to analyze this single matrix.

Idea of the proof. Apply Theorem 6.4.7 to scale an arbitrary representation $[I_3|D]$ into a reduced form. Then use the fact that $\{1,4,7\}, \{2,5,7\}, \{3,6,7\}$ are circuits to force the remaining entries, yielding $D = X$.

From Representations to Graphs

Proposition 6.4.8

Let F be a field. Then

- F_7 is F-representable if and only if the characteristic of F is two;
- F_7^- is F-representable if and only if the characteristic of F is not two.

Idea. By Lemma 6.4.9, every representation can be scaled to $[I_3|X]$. Then Lemma 6.4.4 implies that

$$\mathrm{char}(F) = 2 \Longrightarrow M[I_3|X] = F_7, \qquad \mathrm{char}(F) \neq 2 \Longrightarrow M[I_3|X] = F_7^-.$$

Matroid Representability

Proposition 6.4.10

Let M be a matroid having fewer than eight elements. Then M is representable.

Interpretation. The first non-representable matroids appear on 8 elements.

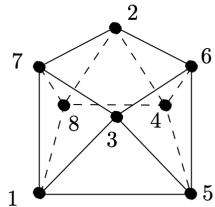
Next example. We now study the family $P_8, P_8^-, P_8^=$, where the construction method from this section leads to sharp field-size and characteristic criteria.

Matroid Representability

Definition of P_8. The matroid P_8 is represented over $GF(3)$ by

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 & 0 \end{array}\right].$$

Matroid Representability



Circuit structure. P_8 has no circuits of size less than four. Its 4-circuits include

$$\{1,2,3,8\},\ \{1,2,4,7\},\ \{1,3,4,6\},\ \{2,3,4,5\},\ \{1,4,5,8\},$$

$$\{2,3,6,7\},\ \{1,5,6,7\},\ \{2,5,6,8\},\ \{3,5,7,8\},\ \{4,6,7,8\}.$$

Among these, $\{1,4,5,8\}$ and $\{2,3,6,7\}$ are the unique pair of disjoint circuit-hyperplanes.

Matroid Representability

Construction from circuit-hyperplane relaxation.

- P_8 has two disjoint circuit-hyperplanes:

$$\{1,4,5,8\}, \qquad \{2,3,6,7\}.$$

- Relaxing exactly one of them gives a matroid denoted P_8^-.
- Relaxing both gives $P_8^=$.

Remark. The two one-step relaxations are isomorphic, so there is only one matroid type called P_8^-.

Matroid Representability

Lemma 6.4.13

Each of the matroids P_8, P_8^-, and $P_8^{=}$ is self-dual. Moreover, both P_8 and $P_8^{=}$ have transitive automorphism groups.

Interpretation. Self-duality lets us translate deletion statements into contraction statements. Transitivity means that all elements are equivalent up to automorphism.

Why this is useful. When proving minor-minimal non-representability later, it is enough to check only one single-element deletion for P_8 or $P_8^{=}$; the other deletions and contractions follow by transitivity and self-duality.

Matroid Representability

Lemma 6.4.14

Suppose that $M \in \{P_8, P_8^-, P_8^=\}$. If M is representable over a field F, then there are elements $b, c \in F - \{0, 1\}$ such that M is represented by $[I_4|D]$, where

$$\begin{array}{c} \\ 1 \\ 4 \\ 7 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 8 & 5 & 6 & 2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & c^{-1} & b \\ 1 & b & 0 & b \\ 1 & c & 1 & 0 \end{bmatrix} \end{array}$$

Idea of the proof. Use Theorem 6.4.7 and the 4-circuits shared by all three matroids to scale an arbitrary representation into a normal form. Then determinant conditions coming from circuits force most entries, leaving only two free parameters b and c.

Matroid Representability

Remaining issue. In the normal form of Lemma 6.4.14, all 4-circuits of $P_8^=$ are present. The only possible *additional* 4-circuits are

$$\{1,4,5,8\} \qquad \text{and} \qquad \{2,3,6,7\}.$$

These occur exactly when

$$\{1,4,5,8\} \text{ is a circuit } \iff b = c,$$

$$\{2,3,6,7\} \text{ is a circuit } \iff b = c^{-1}.$$

Thus, representability reduces to finding $b, c \in F - \{0,1\}$ satisfying certain equations.

Matroid Representability

Lemma 6.4.15

Let F be a field.

(i) P_8 is F-representable if and only if the characteristic of F is not two;

(ii) P_8^- is F-representable if and only if $|F| \geq 4$;

(iii) $P_8^=$ is F-representable if and only if $|F| \geq 5$.

Matroid Representability

Idea of the proof. The two possible extra circuits are controlled by

$$\{1,4,5,8\} \text{ circuit} \iff b = c, \qquad \{2,3,6,7\} \text{ circuit} \iff b = c^{-1}.$$

Thus

$$P_8 : \quad b = c = c^{-1}, \qquad P_8^- : \quad b = c \neq c^{-1}, \qquad P_8^= : \quad b \notin \{c, c^{-1}\}.$$

So representability reduces to asking whether the field contains enough distinct nonzero elements satisfying these relations.

What the Construction Method Achieves

For F_7 and F_7^-. The same partial representation leads to different matroids depending on the characteristic of the field.

For $P_8, P_8^-, P_8^=$. After scaling to a normal form, representability reduces to solving simple equations in field elements b, c.

Takeaway. The construction method turns representability questions into explicit algebra: first determine the pattern $D^\#$, then normalize by scaling, then solve the remaining determinant constraints.