Chapter 16-17.
(0, 1)-Matrices and Latin Squares

A Course in Combinatorics

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(0,1)-Matrices and Line–Sums

- A **(0,1)-matrix** is an $n \times m$ matrix $A = (a_{ij})$ with $a_{ij} \in \{0, 1\}$.
- **Row–sum vector**: $r = (r_1, \ldots, r_n)$ where $r_i = \sum_j a_{ij}$.
- **Column–sum vector**: $s = (s_1, \ldots, s_m)$ where $s_j = \sum_i a_{ij}$.
- Throughout we assume r and s are weakly decreasing partitions of the same integer N; trailing zeros are allowed.

Partitions, Conjugates, Majorization

- **Partition**: weakly decreasing vector of non–negative integers whose sum is N.
- **Conjugate** r^*: in the Ferrers diagram of r, r_k^* equals the number of parts of r that are $\geq k$.

- **Majorization** (r majorizes s, written $r \succeq s$):

$$\sum_{i=1}^{k} r_i \;\geq\; \sum_{i=1}^{k} s_i \quad (\forall k \geq 1)\,.$$

The Averaging Operation

Pick two parts $a \geq b + 2$ and replace (a, b) by $(a - 1, b + 1)$, then reorder the parts. This makes the partition more balanced.

Problem 16A.

Let r and s be partitions of N. Prove

$$r \succeq s \quad \Longleftrightarrow \quad s \text{ can be obtained from } r \text{ by a sequence of the averaging operations.}$$

Existence of (0,1)-matrix

Theorem 16.1.

Let $r = (r_1, \ldots, r_n)$ and $s = (s_1, \ldots, s_m)$ be non–increasing sequences of non–negative integers with $\sum r_i = \sum s_j = N$. There exists an $n \times m$ (0,1)-matrix with row–sum r and column–sum s if and only if the conjugate partition r^* majorizes s.

A Sketch of Proof – Necessity

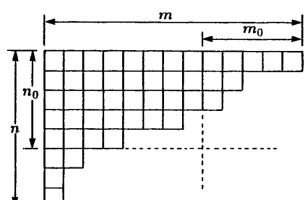
- Consider first k columns; they contain $s_1 + \cdots + s_k$ ones.
- Row i contributes at most $\min(k, r_i)$ ones. Thus

$$\sum_{j=1}^{k} s_j \leq \sum_{i=1}^{n} \min(k, r_i) = \sum_{j=1}^{k} r_j^*.$$

- The equality counts the first k columns of the Ferrers diagram, so $r^* \succeq s$.

A Sketch of Proof – Sufficiency

1. **Max–flow construction**: source $\to$ rows (r_i), unit edges to columns, columns (s_j) $\to$ sink.
2. **Lower bound on cut**: All cuts $\geq N$ iff $r^* \succeq s \Rightarrow$ integral max–flow gives required matrix.



Monotonicity of Counts

Theorem 16.2.

If $r \succeq r_0$ and $s \succeq s_0$ then

$$M(r_0, s_0) \;\geq\; M(r, s),$$

where $M(r, s)$ counts (0,1)-matrices with those line–sums.

A Lower Bound on $A(n, k)$

We will use $A(n, k)$ to denote the number of $n \times n$ $(0, 1)$-matrices with all line-sums k.

Corollary.

$$A(n, k) \geq \frac{\binom{n}{k}^n}{2^{n(k+1)}}$$

A Lower Bound on $A(n, k)$

Problem 16D.

If n is even, show that

$$A(n, \tfrac{n}{2}) \geq \frac{2^{n^2}}{n^{2n}}.$$

Latin Square

A *Latin square* of order n is a quadruple $(R, C, S; L)$ where R, C, and S are sets of cardinality n, and L is a mapping

$$L : R \times C \to S$$

such that for any $i \in R$ and $x \in S$, the equation $L(i, j) = x$ has a unique solution $j \in C$, and for any $j \in C$, $x \in S$, the same equation has a unique solution $i \in R$.

a	b	c	d	e
b	a	e	c	d
c	d	b	e	a
d	e	a	b	c
e	c	d	a	b

Latin Square

a	b	c	d	e
b	a	e	c	d
c	d	b	e	a
d	e	a	b	c
e	c	d	a	b

Problem 17A.

If we fix the first two rows, then there are many ways to fill in the remaining three rows to obtain a Latin square (in fact 24 ways). Show that none of these Latin squares is the multiplication table of a group.

Orthogonal Array OA($n, 3$)

- A **depth-3 orthogonal array** of order n is a $3 \times n^2$ table whose entries are the symbols $1, \ldots, n$.
- **Orthogonality:** In *any two rows* the n^2 vertical pairs are all *different*.
- Notation: $\mathrm{OA}(n, 3)$.

Orthogonal Array OA($n, 3$)

- **Equivalent to a Latin square.**
  Given (r, c, s) rows, the column where $(r_k, c_k) = (i, j)$ gives symbol s_k in cell (i, j).
  The orthogonality guarantees Latin condition, and the process is reversible.

rows	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
columns	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
symbols	3	2	4	1	1	4	2	3	4	3	1	2	2	1	3	4

3	2	4	1
1	4	2	3
4	3	1	2
2	1	3	4

symbols	1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4
columns	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
rows	3	2	4	1	1	4	2	3	4	3	1	2	2	1	3	4

2	4	3	1
4	1	2	3
1	3	4	2
3	2	1	4

Existence of Latin Subsquare

Problem 17D.

Let m and n be positive integers with $m < n$. Show that $m \leq \frac{1}{2}n$ is a necessary and sufficient condition for the existence of a Latin square of order n containing a subsquare of order m.

Partial Latin Squares and Latin Rectangles

- A **partial Latin square** of order n is an $n \times n$ array whose cells are either empty or contain a single symbol from $\{1, \ldots, n\}$, such that *no symbol appears more than once* in any row or column among the filled cells.
- A $k \times n$ **Latin rectangle** $(1 \leq k \leq n)$ is a partial Latin square in which the first k rows are completely filled (each containing all n distinct symbols) and all remaining cells are empty.

1	2	3	4	
				5

1				
	1			
		1		
			1	
				2

Every Latin Rectangle is Fully Extendable

Theorem 17.1.

Let A be a $k \times n$ Latin rectangle with $k < n$. Then A can always be extended to a $(k+1) \times n$ Latin rectangle (and hence, by repeating the argument, completed to a full Latin square of order n).

A Lower Bound on $L(n)$

Theorem 17.2.

For every positive integer n, the number $L(n)$ of Latin squares of order n satisfies

$$L(n) \geq \frac{(n!)^{2n}}{n^{n^2}}.$$

Completion Criterion for Partial Latin Squares

Theorem 17.4.

Let A be a partial Latin square of order n whose filled cells form an $r \times s$ sub-array, i.e. cell (i, j) is filled *iff* $1 \leq i \leq r$ and $1 \leq j \leq s$. Denote by $N(i)$ the number of entries of A equal to the symbol i. Then A can be completed to a Latin square of order n if and only if

$$N(i) \geq r + s - n \quad \text{for every } i \in \{1, \ldots, n\}.$$

($\Rightarrow$) In any Latin square of order n the first r rows contain exactly r copies of each symbol i. Of these at most $n - s$ can lie in the last $n - s$ columns, so at least $r - (n - s) = r + s - n$ must appear in the filled $r \times s$ block. Hence $N(i) \geq r + s - n$.

Completion Criterion for Partial Latin Squares

($\Leftarrow$) Define the $r \times n$ $(0,1)$–matrix $B = [b_{ij}]$ by

$$b_{ij} = 1 \iff \text{symbol } j \text{ is } \textit{not} \text{ present in row } i \text{ of } A.$$

- Each row of B has exactly $n - s =: d$ ones (the yet-unfilled cells in that row).
- Column j has $r - N(j) \leq d$ ones by the hypothesis $N(j) \geq r + s - n$.

If $d = 0$, A is already a Latin rectangle. For $d > 0$, Theorem 7.5(ii) splits an integral matrix into d integral matrices whose entries and line-sums round those of B/d up or down. Here this gives

$$B = L(s{+}1) \; + \; \cdots \; + \; L(n),$$

where, for each $t = s + 1, \ldots, n$, $L(t) = [\ell_{ij}^{(t)}]$ is an $r \times n$ $(0,1)$–matrix having *one* 1 in every row and *at most* one 1 in every column.

Completion Criterion for Partial Latin Squares

For every $i \in \{1, \dots, r\}$ and every $t \in \{s+1, \dots, n\}$ locate the unique index k such that $\ell_{ik}^{(t)} = 1$, and write k into cell (i, t) of A. Each row of $L(t)$ contains exactly one 1, so each new cell is filled once. Each column contains at most one 1, so no symbol repeats within the new column. Since $\sum_t L(t) = B$, each missing symbol is used exactly once in its row, and no previously present symbol is used again. Thus the Latin property is preserved and the first r rows become completely filled, giving a $k = r$ Latin rectangle.

Finally, by Theorem 17.1 a $k \times n$ Latin rectangle can always be extended one row at a time until an $n \times n$ Latin square is obtained. Hence the stated condition is sufficient. $\square$