6. Representable Matroids

6.10. Dowling geometries

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Based on *Matroid Theory* by James Oxley

Apr 27, 2026

Dowling Geometries

Motivation. Over a field $\mathrm{GF}(q)$, one may start with a basis $\{v_1, v_2, \ldots, v_r\}$ of $V(r, q)$ and consider the points

$$\{v_1, \ldots, v_r\} \cup \{\, v_i - \alpha v_j : 1 \le i < j \le r,\ \alpha \in \mathrm{GF}(q)^\times\}.$$

The matroid induced on this set is denoted by $Q_r(\mathrm{GF}(q)^\times)$.

Special cases.

$$Q_2(\mathrm{GF}(q)^\times) \cong U_{2,q+1}, \qquad Q_r(\mathrm{GF}(2)^\times) \cong M(K_{r+1}).$$

Thus Dowling geometries connect naturally to both uniform matroids and graphic matroids.

Dowling Geometries

General definition. Let A be a finite multiplicative group with identity 1, and let $r \geq 0$. The Dowling geometry $Q_r(A)$ is defined from a labelled graph K_r^A.

Construction of K_r^A. For $r \geq 1$, the vertex set is $\{1, 2, \ldots, r\}$, and:

(i) for each pair $i < j$ and each $\alpha \in A$, add an edge joining i and j, labelled α_{ij};

(ii) for each vertex i, add a loop labelled b_i.

For convenience, when $j < i$, we write $\alpha_{ij} = (\alpha^{-1})_{ji}$.

Terminology. The elements $b_1, \ldots, b_r$ are called the **joints** of $Q_r(A)$.

Dowling Geometries

Balanced cycles. Let C be a cycle of K_r^A with at least two edges, and choose an orientation for C. If an edge e_j has label β, define

$$\phi(e_j) = \begin{cases} \beta, & \text{if it is traversed from the smaller vertex to the larger one,} \\ \beta^{-1}, & \text{otherwise.} \end{cases}$$

Then C is called **balanced** if

$$\phi(e_1)\phi(e_2)\cdots\phi(e_k) = 1.$$

Dowling Geometries

Unbalanced cycles. A cycle is **unbalanced** if either it is a single loop, or it has at least two edges but the above product is not equal to 1.

Key point. This notion is well-defined: it does not depend on the chosen starting vertex or the chosen orientation of the cycle.

Dowling Geometries

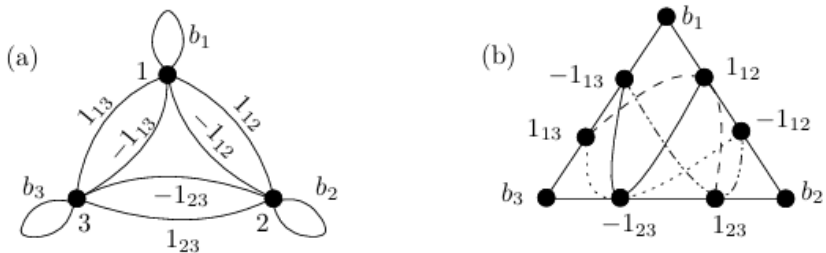
A Θ-graph consists of three internally disjoint paths with the same endpoints. A tight handcuff has two cycles sharing one vertex; a loose handcuff has two disjoint cycles joined by a path. Here, *unbalanced* means that none of the cycles is balanced.

Circuits of $Q_r(A)$. They consist of

- the edge sets of all balanced cycles;
- the edge sets of all unbalanced Θ-graphs;
- the edge sets of all unbalanced loose handcuffs;
- the edge sets of all unbalanced tight handcuffs.

Dowling Geometries

A first concrete example. When $A = \{1, -1\}$, the graph K_3^A has four balanced 3-cycles. In the geometric representation of $Q_3(A)$, these correspond to four additional 3-point lines.



Biased Graphs

Why biased graphs appear. For $Q_r(A)$, we have already described certain subsets that should be the circuits: balanced cycles, and unbalanced Θ-graphs and handcuffs. What remains is to show that these subsets really satisfy the circuit axioms of a matroid.

Definition. A biased graph (G, Ψ) consists of a graph G together with a set Ψ of cycles of G such that if $C_1, C_2 \in \Psi$ and $G[C_1 \cup C_2]$ is a Θ-graph, then the third cycle in $G[C_1 \cup C_2]$ also belongs to Ψ.

Strategy. We first show that the balanced cycles of K_r^A satisfy this property, and then apply a general theorem producing a matroid from a biased graph.

Biased Graphs

Lemma 6.10.3

In K_r^A, if two of the cycles in a Θ-subgraph are balanced, then so is the third.

Idea of the proof. Traverse all three paths from the same endpoint u to the other endpoint v, and let $\delta_1, \delta_2, \delta_3$ be their adjusted label products. A cycle uses one path forward and another backward. If two cycles are balanced,

$$\delta_1 \delta_2^{-1} = 1 \qquad \text{and} \qquad \delta_1 \delta_3^{-1} = 1.$$

Thus $\delta_1 = \delta_2 = \delta_3$, so $\delta_2 \delta_3^{-1} = 1$: the third cycle is balanced as well.

Interpretation. Thus the balanced cycles of K_r^A satisfy the defining closure property of a biased graph.

Bias Matroids

Theorem 6.10.5

Let (G, Ψ) be a biased graph. Then there is a matroid on $E(G)$ whose collection C of circuits consists of the edge sets of all balanced cycles along with all unbalanced Θ-graphs, all unbalanced loose handcuffs, and all unbalanced tight handcuffs.

Terminology. The matroid given by this theorem is called the **bias matroid** or **frame matroid** of (G, Ψ), and is denoted by $M(G, \Psi)$.

Why this matters. This theorem is exactly what allows us to conclude that the circuit description for $Q_r(A)$ really defines a matroid.

Bias Matroids

Interpretation. Theorem 6.10.5 unifies several familiar matroids in a single construction.

- If Ψ is the set of all cycles of G, then

$$M(G, \Psi) = M(G),$$

  the usual cycle matroid.
- If $\Psi = \varnothing$, then $M(G, \Psi)$ is the **bicircular matroid** $B(G)$, whose circuits are precisely the Θ-graphs, loose handcuffs, and tight handcuffs.

Message. So bias matroids are not an isolated construction; they provide a common framework containing several important examples.

Bias Matroids

Idea of the proof of Theorem 6.10.5. The only substantial point is the circuit elimination axiom. So take two distinct candidate circuits C_1 and C_2 with $e \in C_1 \cap C_2$, and consider

$$H = G[C_1 \cup C_2] \setminus e.$$

Using Lemma 6.10.4, any connected graph with at least two cycles contains a Θ-graph, a loose handcuff, or a tight handcuff. One then analyzes the possible shape of H and shows that $(C_1 \cup C_2) - e$ must contain another member of the proposed circuit family.

Return to $Q_r(A)$. By Lemma 6.10.3, the balanced cycles of K_r^A form a biased graph. Therefore Theorem 6.10.5 implies that $Q_r(A)$ is indeed a matroid.

Bias Matroids

Proposition 6.10.7

The class of bias matroids is minor-closed.

Interpretation. This shows that bias matroids form a robust class: they are closed under both deletion and contraction.

Dowling Geometries

Proposition 6.10.8

(i) For all groups A,

$$Q_0(A) = U_{0,0}, \qquad Q_1(A) \cong U_{1,1}, \qquad Q_2(A) \cong U_{2,|A|+2}.$$

(ii) For $r \geq 3$, if $Q_r(A) \cong Q_r(A')$ for groups A and A', then $A \cong A'$.

Interpretation. For small ranks, $Q_r(A)$ is essentially uniform. But from rank 3 onward, the geometry is rigid enough to recover the underlying group.

Dowling Geometries

Lemma 6.10.9

Let A and A' be groups, and let $\phi_1, \phi_2, \phi_3 : A \to A'$ be bijections. If

$$\phi_3(\alpha\beta) = \phi_1(\alpha)\phi_2(\beta) \qquad \text{for all } \alpha, \beta \in A,$$

then $A \cong A'$.

Dowling Geometries

Idea of the proof of Proposition 6.10.8. If $|A| < 4$, its order determines the group. Otherwise the joints can be recognized from their incident lines of size $|A| + 2$. After aligning three joints under an isomorphism, restrict to their rank-3 flat. The three families

$$\{\alpha_{12}\}, \qquad \{\beta_{23}\}, \qquad \{\gamma_{13}\}$$

encode multiplication through collinearity:

$$\alpha\beta = \gamma \quad \Longleftrightarrow \quad \{\alpha_{12}, \beta_{23}, \gamma_{13}\} \text{ is a line in } Q_3(A).$$

Hence an isomorphism $Q_3(A) \cong Q_3(A')$ induces bijections satisfying the hypothesis of Lemma 6.10.9, so $A \cong A'$.

Representability of Dowling Geometries

Theorem 6.10.10

Suppose $r \geq 3$. Then $Q_r(A)$ is representable over a field F if and only if A is isomorphic to a subgroup of $F^\times$.

Interpretation. The representability problem for $Q_r(A)$ is completely controlled by group theory: $Q_r(A)$ is representable over F exactly when the multiplication of A can be realized inside the multiplicative group of the field.

Remark. The interesting case is $r \geq 3$. For $r = 1$, the matroid is $U_{1,1}$, and for $r = 2$, it is $U_{2,|A|+2}$.

Representability of Dowling Geometries

Lemma 6.10.11

Let F be a field and let A be a finite subgroup of $F^\times$. For $r \geq 1$, let $B = \{v_1, v_2, \ldots, v_r\}$ be a basis of $V(r, F)$, and let M be the matroid induced on

$$B \cup \{\, v_i - \alpha v_j : 1 \leq i < j \leq r,\ \alpha \in A \,\}$$

by linear independence over F. Then the map sending

$$v_i \mapsto b_i, \qquad v_i - \alpha v_j \mapsto \alpha_{ij}$$

is an isomorphism between M and $Q_r(A)$.

Interpretation. This gives the easy direction of Theorem 6.10.10. If $A \leq F^\times$, then the earlier field-based construction really does produce $Q_r(A)$.

Representability of Dowling Geometries

Idea of the converse direction. Assume $Q_r(A)$ is representable over F. Since $Q_3(A)$ is a restriction of $Q_r(A)$, it suffices to work in rank 3.

Choose the joint columns b_1, b_2, b_3 as basis vectors v_1, v_2, v_3. The edge α_{ij} lies on the line through b_i, b_j and is parallel to neither, so nonzero column scaling puts it in the form

$$v_i - \sigma_{ij}(\alpha)v_j, \qquad \sigma_{ij}(\alpha) \in F^{\times}.$$

Distinct edge elements give distinct values of σ_{ij}. Then the circuit relation in $Q_3(A)$ implies

$$\alpha\beta\gamma = 1 \quad \Longleftrightarrow \quad \sigma_{12}(\alpha)\sigma_{23}(\beta)\sigma_{31}(\gamma) = 1.$$

Representability of Dowling Geometries

Normalizing the images. Let $H_3 = \sigma_{12}(A)$, $H_1 = \sigma_{23}(A)$, and $H_2 = \sigma_{31}(A)$. The circuit relations imply

$$u, v, w \in H_i \quad \Longrightarrow \quad uvw^{-1} \in H_i.$$

For a fixed $c_i \in H_i$, this makes H_i/c_i a subgroup of $F^\times$. The same relations show that these three subgroups coincide, say with A_0.

Rescaling the three maps gives bijections $\tau_{12}, \tau_{23}, \tau_{13} : A \to A_0$ satisfying

$$\tau_{13}(\alpha\beta) = \tau_{12}(\alpha)\tau_{23}(\beta).$$

Lemma 6.10.9 now gives $A \cong A_0 \leq F^\times$.