Chapter 23.
Projective and combinatorial geometries

A Course in Combinatorics

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Combinatorial geometry

A *combinatorial geometry* is a pair $(X, \mathcal{F})$, where X is a set of points and $\mathcal{F}$ is a family of subsets of X called *flats*, such that

1. $\mathcal{F}$ is closed under (pairwise) intersection,
2. there are no infinite chains in the poset $\mathcal{F}$,
3. $\mathcal{F}$ contains the empty set, all singletons $\{x\}$ for $x \in X$, and the set X itself,
4. for every flat $E \in \mathcal{F}$ with $E \neq X$, the sets $F \setminus E$, as F ranges over the flats covering E, partition $X \setminus E$.

Here, we say that $F \in \mathcal{F}$ *covers* $E \in \mathcal{F}$ if $E \subset F$, but there does not exist $G \in \mathcal{F}$ such that $E \subset G \subset F$.

This property is familiar from geometry: lines through a point, with that point removed, partition the remaining points; planes through a line, with that line removed, do the same.

Combinatorial geometry

Example (23.1)

Every linear space (as introduced in Chapter 19) gives us a combinatorial geometry on its point set X when we take as flats

$$\emptyset, \quad \{x\}\ (x \in X), \quad \text{all lines, and } X \text{ itself.}$$

The fact that the lines through a given point, with that point removed, partition the remaining points is another way of saying that two points determine a unique line.

Combinatorial geometry

Example (23.2)

Every Steiner system $S(t, k, v)$ with $t \geq 2$ gives us a combinatorial geometry on its point set X when we take as flats

$$\{ E \subseteq X : |E| < t \}, \quad \text{all blocks, and } X.$$

To get a geometry by this construction, it is not necessary that all blocks have the same size, but only that each t-subset is contained in a unique block.

Combinatorial geometry

Example (23.3)

Let V be an n-dimensional vector space over a field $\mathbb{F}$. By an *affine subspace* of V, we mean either the empty set or a coset (or translate) of a linear subspace of V in the additive group.
For example, for $a, b \in \mathbb{F}$, the subset

$$\{(x, y) \in \mathbb{F}^2 : y = ax + b\}$$

is an affine subspace of $\mathbb{F}^2$.
The set V, together with all affine subspaces, forms a combinatorial geometry called the *affine geometry* $\mathrm{AG}_n(\mathbb{F})$. We write $\mathrm{AG}_n(q)$ for $\mathrm{AG}_n(\mathbb{F}_q)$; the case $n = 2$ was introduced in Chapter 19.

Combinatorial geometry

Example (23.4)

The projective geometry $PG_n(\mathbb{F})$, while more fundamental than $AG_n(\mathbb{F})$, can be a little more awkward to define.
Let V be an $(n+1)$-dimensional vector space over a field $\mathbb{F}$. The point set X of $PG_n(\mathbb{F})$ consists of all the 1-dimensional (linear) subspaces of V.
For example, if $\mathbb{F} = \mathbb{F}_q$ is the finite field of q elements, then the number of projective points of $PG_n(q)$ is

$$\frac{q^{n+1}-1}{q-1} = q^n + q^{n-1} + \cdots + q^2 + q + 1.$$

Combinatorial geometry

Example (23.4)

To each linear subspace $W \subseteq V$, we associate a flat

$$F_W = \{\, U \subseteq V : U \text{ is a 1-dimensional subspace with } U \subseteq W \,\}.$$

That is, F_W consists of all the 1-dimensional subspaces of V that are contained in W. Let $\mathcal{F}$ be the set of all such flats F_W. Then the projective geometry $PG_n(\mathbb{F})$ is the pair $(X, \mathcal{F})$.

Subgeometry

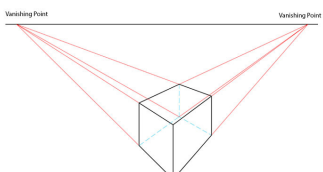
Let Y be any subset of the elements X of a combinatorial geometry $(X, \mathcal{F})$ and let

$$\mathcal{E} := \{ F \cap Y : F \in \mathcal{F} \}.$$

Then $(Y, \mathcal{E})$ is also a combinatorial geometry, the *subgeometry* on Y.

Subgeometry

For example, $AG_n(\mathbb{F})$ is a subgeometry of $PG_n(\mathbb{F})$. There is a standard embedding of the point set of $AG_n(\mathbb{F})$ into the point set of $PG_n(\mathbb{F})$: to each vector x in an n-dimensional vector space V, associate the 1-dimensional subspace that is the span of $(x, 1)$ in the $(n+1)$-dimensional space $V \times \mathbb{F}$.
The image consists of all projective points not contained in the hyperplane $V \times \{0\}$.



Lattice

A lattice L is a partially ordered set with the property that any nonempty finite subset $S \subseteq L$ has a meet (or greatest lower bound), that is, an element $b \in L$ so that

$$\forall a \in S \, [\, b \leq a \,] \quad \text{and} \quad \forall a \in S \, [\, c \leq a \,] \quad \Rightarrow \quad c \leq b,$$

as well as a join (or least upper bound), that is an element $b \in L$ so that

$$\forall a \in S \, [\, b \geq a \,] \quad \text{and} \quad \forall a \in S \, [\, c \geq a \,] \quad \Rightarrow \quad c \geq b.$$

Lattice

The meet and join of a two-element set $S = \{x, y\}$ are denoted, respectively, by $x \wedge y$ and $x \vee y$. It is easily seen that $\wedge$ and $\vee$ are commutative, associative, idempotent binary operations; moreover, if all two-element subsets have meets and joins, then any finite nonempty subset has a meet and a join.
The lattices we will consider have the property that there are no infinite chains. Such a lattice has a (unique) least element (that we denote by 0_L) because the condition that no infinite chains exist allows us to find a minimal element m, and any minimal element is minimum since if $m \not\leq a$, then $m \wedge a$ would be less than m. Similarly, there is a unique largest element 1_L.

Poset Terminology

For elements a and b of a poset, we say that a *covers* b and write

$$a \gtrdot b$$

when $a > b$ but there are no elements c so that $a > c > b$.
For example, when U and W are linear subspaces of a vector space, we have

$$U \gtrdot W \quad \text{when } U \supset W \text{ and } \dim(U) = \dim(W) + 1.$$

Recall that a *chain* in a partially ordered set P is a totally ordered subset of P. So a finite chain can be thought of as a sequence

$$a_0 < a_1 < \cdots < a_n.$$

A *point* of a lattice L with minimum element 0_L is an element that covers 0_L.

Geometric lattice

A *geometric lattice* is a lattice L that has no infinite chains, and such that

1. L is *atomic*, that is, each element of L is the join of points of L, and
2. L is *semimodular*, that is, if a and b are distinct and both cover c in L, then $a \vee b$ covers both a and b.

Geometric lattice

Theorem 23.1.

The set of flats of a combinatorial geometry, ordered by inclusion, is a geometric lattice. Conversely, given a geometric lattice L with points X, then $(X, \{F_y : y \in L\})$ is a combinatorial geometry, where $F_y := \{x \in X : x \leq y\}$.

Proof. Since $\mathcal{F}$ is closed under intersection and contains X and all singletons, the poset of flats is an atomic lattice (the meet of two flats is their intersection; the join of two flats is the intersection of all flats containing both).

Geometric lattice

Suppose that distinct flats F_1 and F_2 cover a flat E in a combinatorial geometry. Let x be a point in $F_1 \setminus F_2$. There is a flat G_2 that covers F_2 in $\mathcal{F}$ and contains the point x. G_2 contains F_1 since otherwise $E \subset F_1 \cap G_2 \subset F_1$. By a symmetric argument, there is a flat G_1 that covers F_1 and also contains F_1 and F_2. The join $F_1 \vee F_2$ must be contained in both G_1 and G_2. Since G_i covers F_i, we conclude $F_1 \vee F_2 = G_1 = G_2$.

Geometric lattice

Now let L be a geometric lattice and define the flats F_y as subsets of the point set X of L as in the statement of the theorem. It is clear that the empty set, any singleton, and X are flats (when y is taken to be 0_L, a point of L, or 1_L, respectively).
Since $x \leq y$ and $x \leq z$ if and only if $x \leq y \wedge z$, we have

$$F_y \cap F_z = F_{y \wedge z}.$$

Thus $\mathcal{F} := \{F_y : y \in L\}$ is closed under intersection. Atomicity gives $F_y \subseteq F_z$ if and only if $y \leq z$, so this family has no infinite chains.

Geometric lattice

No point $x \notin F_y$ can be in two flats that cover F_y, so it remains only to show that some flat that covers F_y contains x. We show that $F_{x \vee y}$ covers F_y to complete the proof. This is equivalent to showing that $x \vee y \succcurlyeq y$ in L whenever $x \not\leq y$ and $x \succcurlyeq 0_L$ is a point of L. If $y = 0_L$, this is immediate. Otherwise, choose a maximal chain

$$0_L < y_1 < y_2 < \cdots < y_k = y.$$

Since x and y_1 cover 0_L, $x \vee y_1$ covers y_1. Since $x \vee y_1$ and y_2 cover y_1, $x \vee y_2$ covers y_2 (clearly $x \vee y_1 \vee y_2 = x \vee y_2$). Inductively, we find that $x \vee y_k$ covers y_k. □

Geometric lattice

Problem 23A.

(i) Any interval of $PG_n(\mathbb{F})$ is isomorphic as a poset to $PG_m(\mathbb{F})$ for some $m \leq n$.

(ii) $PG_n(\mathbb{F})$ is isomorphic to its dual poset (where the order is reversed).

Partition lattices

Example (23.5)

The partition lattices Π_n, whose elements are all partitions of an n-set X, provide another family of geometric lattices, and hence give us combinatorial geometries. The partitions are ordered by refinement; A is a refinement of B if each block of B is the union of blocks of A.

So the least element (the finest partition) is all singletons

$$\{\{x\} : x \in X\},$$

and the greatest element is $\{X\}$, consisting of a single block.

A moment's thought shows that one partition covers another if the former is obtained by coalescing two blocks of the latter into one. The points are the partitions that consist of a single block of size 2 and $n-2$ singletons.

Partition lattices

Example (23.5)

For example, Π_4 has 15 elements:

$$\{1234\},$$

$$\{123,4\},\{124,3\},\{134,2\},\{234,1\},\{12,34\},\{13,24\},\{14,23\},$$

$$\{12,3,4\},\{13,2,4\},\{14,2,3\},\{23,1,4\},\{24,1,3\},\{34,1,2\},$$

$$\{1,2,3,4\}.$$

To simplify the notation, we have written, for example, $\{123,4\}$ for what should be $\{\{1,2,3\},\{4\}\}$.

Partition lattices

Example (23.6)

The points of Π_n are in one-to-one correspondence with the edges of K_n. Given any simple graph G on n vertices, we obtain a geometric lattice $L(G)$ whose points correspond to the edge set $E(G)$ of G as follows.
The elements of $L(G)$ are all those partitions A of $V(G)$ such that the subgraph of G induced by each block of A is connected. The partitions with this property are exactly those that are the join (in Π_n) of points (in Π_n) corresponding to some subset of the edges of G.
The lattices $L(G)$ are sometimes called the *lattices of contractions* of G (see Chapter 33).